Exercises for Stochastic Processes

Tutorial exercises:

T1. Let $T \subseteq \mathbb{R}$. Show that a stochastic process $\mathbb{X} := (X_t)_{t\in T}$ on $(\Omega, \mathcal{F})$ with values in S^T is $\mathcal{F} - \mathcal{S}^T$-measurable if and only if all projections X_t are $\mathcal{F} - \mathcal{S}$-measurable. ($\mathcal{S}$ denotes a σ-algebra on S.)

T2. Let $(X_t)_{t\in\mathbb{R}}$ be a real-valued $\mathcal{F} - \mathcal{B}^{\mathbb{R}}$-measurable stochastic process with continuous paths. Show that $\sup_{t\in\mathbb{R}} X_t$ is measurable.

T3. Let $\tau_1, \tau_2, \dots$ be independent and exponentially distributed with parameter $\lambda > 0$. Define

$$N_t := \left|\{k \geq 1 \mid \tau_1 + \cdots + \tau_k \leq t\}\right| .$$

Show that, if $0 < s < t$, then N_s and $N_t - N_s$ are independently Poisson distributed with parameters λs and $\lambda(t - s)$.

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Homework exercises:

H1. (a) Let $(S, \mathcal{S})$ be a measurable space. Show that, for uncountable $T \subset \mathbb{R}$,

$$\mathcal{S}^T = \left\{ \{ f \in S^T \mid (f(t_1), f(t_2), \dots) \in A \} \mid t_1, t_2, \dots \in T, A \in \mathcal{S}^{\{t_1, t_2, \dots\}} \right\}.$$

("All sets in the product σ-algebra are countably determined.")

(b) Conclude that the set of all continuous functions on $T \subset \mathbb{R}$ is no (product-)measurable subset of $\mathbb{R}^T$.

H2. Deduce Kolmogorov's continuity criterion (Theorem 2.1 in the lecture) from Theorem 2.2.

H3. Under which (necessary and sufficient) condition does an i.i.d. family $(X_t)_{t \in \mathbb{R}}$ have a continuous modification?

Deadline: Monday, 21.10.19